Projective maximal families of orthogonal measures with large continuum

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We study maximal orthogonal families of Borel probability measures on 2^ω (abbreviated m.o. families) and show that there are generic extensions of the constructible universe L in which each of the following holds:

(1) There is a Δ^1_3-definable well order of the reals, there is a Π^1_2-definable m.o. family, there are no $\mathbf{\Sigma}^1_2$-definable m.o. families and $\mathfrak{b} = \mathfrak{c} = \omega_3$ (in fact any reasonable value of $\mathfrak{c}$ will do).

(2) There is a Δ^1_3-definable well order of the reals, there is a Π^1_2-definable m.o. family, there are no $\mathbf{\Sigma}^1_2$-definable m.o. families, $\mathfrak{b} = \omega_1$ and $\mathfrak{c} = \omega_2$.

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1 Introduction

Let X be a Polish space, and let $P(X)$ denote the Polish space of Borel probability measures on X, in the sense of [9, 17.E]. Recall that if $\mu, \nu \in P(X)$ then μ and ν are said to be *orthogonal*, written $\mu \perp \nu$, if there is a Borel set $B \subseteq X$ such that $\mu(B) = 0$ and $\nu(X \setminus B) = 0$. A set of measures $\mathcal{A} \subseteq P(X)$ is said to be *orthogonal* if whenever $\mu, \nu \in \mathcal{A}$ and $\mu \neq \nu$ then $\mu \perp \nu$. A *maximal orthogonal family*, or *m.o. family*, is an orthogonal family $\mathcal{A} \subseteq P(X)$ which is maximal under inclusion.

The present paper is concerned with the study of *definable* m.o. families. A well-known result to Preiss and Rataj [13] states that there are no analytic m.o. families, and in a recent paper [3] it was shown by Fischer and Törnquist that if all reals are constructible then there is a Π^1_1 m.o. family. The latter paper also raised the question how restrictive the existence of a definable m.o. family is on the structure of the real line, since it was shown that Π^1_1 m.o. families cannot coexist with Cohen reals.

In the present paper we study Π^1_2 m.o. families in the context of $\mathfrak{c} \geq \omega_2$, with the additional requirement that there is a Δ^1_3-definable wellorder of $\mathbb{R}$. Our main results are:

Theorem 1 It is consistent with $\mathfrak{c} = \mathfrak{b} = \omega_3$ that there is a Δ^1_3-definable wellorder of the reals, a Π^1_2 definable maximal orthogonal family of measures and there are no $\mathbf{\Sigma}^1_2$-definable maximal sets of orthogonal measures.

There is nothing special about $\mathfrak{c} = \omega_3$. In fact the same result can be obtained for any reasonable value of $\mathfrak{c}$.

Theorem 2 It is consistent with $\mathfrak{b} = \omega_1$, $\mathfrak{c} = \omega_2$ that there is a Δ^1_3-definable wellorder of the reals, a Π^1_2 definable maximal orthogonal family of measures and there are no $\mathbf{\Sigma}^1_2$-definable maximal sets of orthogonal measures.

Taken together these theorems indicate that the existence of a Π^1_2 m.o. family does not seem to impose any severe restrictions on the structure of the real line. On the other hand, we show (Proposition 1) that Σ^1_2 m.o. families cannot coexist with either Cohen or random reals, which is why in the models produced to prove Theorems 1 and 2 there are no $\mathbf{\Sigma}^1_2$ m.o. families.

The theorems of this paper belong to a line of results concerning the definability of certain combinatorial objects on the real line and in particular the question of how low in the projective hierarchy such objects exist. In [12] Mathias showed that there is no Σ^1_1-definable maximal almost disjoint (mad) family in $[\omega]^\omega$. Assuming $V = L$, Miller obtained (see [11]) a Π^1_1 mad family in $[\omega]^\omega$.

The study of the existence of definable combinatorial objects on $\mathbb{R}$ in the presence of a projective wellorder of the reals and $\mathfrak{c} \geq \omega_2$ was initiated in [1], [4] and [2]. The wellorder of $\mathbb{R}$ in all those models has a Δ^1_3-definition, which is indeed optimal for models of $\mathfrak{c} \geq \omega_2$, since by Mansfield's theorem (see [7, Theorem 25.39]) the existence of a Σ^1_2-definable wellorder of the reals implies that all reals are constructible. The existence of a Π^1_2-definable ω-mad family in $[\omega]^\omega$ in the presence of $\mathfrak{c} = \mathfrak{b} = \omega_2$ was established by Friedman and Zdomskyy in [4]. In the same paper, referring to earlier results (see [14] and [8]) they outlined the construction of a model in which $\mathfrak{c} = \omega_2$ and there is a Π^1_1-definable ω-mad family: Start with the constructible universe L, obtain a Π^1_1-definable ω mad family and proceed with a countable support iteration of length ω_2 of Miller forcing. The techniques were further developed in [2] to establish a model in which there is a Π^1_2-definable ω-mad family and $\mathfrak{c} = \mathfrak{b} = \omega_3$. In particular, in the models from [4] and [2], there are no maximal almost disjoint families of size $< \mathfrak{c}$ and so the almost disjointness number has a Π^1_2-witness.

The present paper combines the encoding techniques of [3] with the techniques of [1, 4, 2] to obtain Theorems 1 and 2. We note that one significant difference from the situation for mad families is that m.o. families always have size $\mathfrak{c}$ (see [3, Proposition 4.1]).

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2 Preliminaries

In this section, we briefly recall the coding of probability measures on 2^ω and the encoding technique for measures introduced in [3].

Let X be a Polish space. Recall that measures if $\mu, \nu \in P(X)$ then μ is said to be *absolutely continuous* with respect to ν, written $\mu \ll \nu$, if for all Borel subsets of X we have that $\nu(B) = 0$ implies that $\mu(B) = 0$. Two measures $\mu, \nu \in P(2^\omega)$ are called *absolutely equivalent*, written $\mu \approx \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.

If $s \in 2^{<\omega}$ we let $N_s = \{x \in 2^\omega : s \subseteq x\}$ be the basic neighbourhood determined by s. Following [3], we let

$$p(2^\omega) = \{f : 2^{<\omega} \to [0,1] : f(\emptyset) = 1 \wedge (\forall s \in 2^{<\omega}) f(s) = f(s^\frown 0) + f(s^\frown 1)\}.$$

The spaces $p(2^\omega)$ and $P(2^\omega)$ are homeomorphic via the recursively defined isomorphism $f \mapsto \mu_f$ where $\mu_f \in P(2^\omega)$ is the measure uniquely determined by requiring that $\mu_f(N_s) = f(s)$ for all $s \in 2^{<\omega}$. We call the unique real $f \in p(2^\omega)$ such that $\mu = \mu_f$ the *code* for μ. The identification of $P(2^\omega)$ and $p(2^\omega)$ allow us to use the notions of effective descriptive set theory in the space $P(2^\omega)$. For instance, the set $P_c(2^\omega)$ of all non-atomic probability measures on 2^ω is arithmetical because the set $p_c(2^\omega) = \{f \in p(2^\omega) : \mu_f \text{ is non-atomic}\}$ is easily seen to be arithmetical, as shown in [3].

We will use the method of coding a real $z \in 2^\omega$ into a measure $\mu \in P_c(2^\omega)$ introduced in [3]. For convenience we repeat the construction in minimal detail. Given $\mu \in P_c(2^\omega)$ and $s \in 2^{<\omega}$ we let $t(s, \mu)$ be the lexicographically least $t \in 2^{<\omega}$ such that $s \subseteq t$, $\mu(N_{t^\frown 0}) > 0$ and $\mu(N_{t^\frown 1}) > 0$, if it exists and otherwise we let $t(s, \mu) = \emptyset$. Define recursively $t_n^\mu \in 2^{<\omega}$ by letting $t_0^\mu = \emptyset$ and $t_{n+1}^\mu = t(t_n^{\mu\frown} 0, \mu)$. Since μ is non-atomic, we have $\mathrm{lh}(t_{n+1}^\mu) > \mathrm{lh}(t_n^\mu)$. Let $t_\infty^\mu = \bigcup_{n=0}^\infty t_n^\mu$. For $f \in p_c(2^\omega)$ and $n \in \omega \cup \{\infty\}$ we will write t_n^f for $t_n^{\mu_f}$. Clearly the sequence $(t_n^f : n \in \omega)$ is recursive in f.

Define the relation $R \subseteq p_c(2^\omega) \times 2^\omega$ as follows:

$$\begin{aligned} R(f,z) \iff (\forall n \in \omega)\big(z(n) = 1 \longleftrightarrow (f(t_n^{f\frown} 0) = \tfrac{2}{3} f(t_n^f) \wedge f(t_n^\frown 1) = \tfrac{1}{3} f(t_n))\big) \\ \wedge\big(z(n) = 0 \leftrightarrow f(t_n^{f\frown} 0) = \tfrac{1}{3} f(t_n^f) \wedge f(t_n^{f\frown} 1) = \tfrac{2}{3} f(t_n^f)\big). \end{aligned}$$

Whenever $(f, z) \in R$ we say that f *codes* z. Note that $\mathrm{dom}(R) = \{f \in p_c(2^\omega) : (\exists z) R(f, z)\}$ is Π_1^0 and so the function $r : \mathrm{dom}(R) \to 2^\omega$, where $r(f) = z$ if and only if $(f, z) \in R$, is also Π_1^0. If ν is a measure such that $\nu = \mu_f$ for some code f, then let $r(\nu) = r(f)$. The key properties of this construction is contained in the following Lemma (see [3, Coding Lemma]):

Lemma 1 There is a recursive function $G : p_c(2^\omega) \times 2^\omega \to p_c(2^\omega)$ such that $\mu_{G(f,z)} \approx \mu_f$ and $R(G(f, z), z)$ for all $f \in p_c(2^\omega)$ and $z \in 2^\omega$.

The proofs of Theorems 1 and 2 use the following result, which we now prove.

Proposition 1 Let $a \in \mathbb{R}$ and suppose that there either is a Cohen real over $L[a]$ or there is a random real over $L[a]$. Then there is no $\Sigma^1_2(a)$ m.o. family.

We first need a preparatory Lemma. In 2^ω, consider the equivalence E_I defined by

$$xE_Iy \iff \sum_{n=0}^{\infty} \frac{|x(n)-y(n)|}{n+1} < \infty.$$

We identify 2^ω with $\mathbb{Z}_2^\omega$ and equip it with the Haar measure μ.

Lemma 2 Let $A \subseteq 2^\omega$ be a Borel set such that $\mu(A) > 0$. Then $E_I \leq_B E_I \restriction A$, where $E_I \restriction A$ is the restriction of E_I to A.

Notation: The constant 0 sequence of length $n \in \omega \cup \{\infty\}$ is denoted 0^n. If $A \subseteq 2^\omega$ and $s \in 2^{<\omega}$ let

$$A_{(s)} = \{x \in 2^\omega : s^\frown x \in A\},$$

the *localization* of A at s.

Proof of Lemma 2 Without loss of generality assume that $A \subseteq 2^\omega$ is closed. We will define $q_n \in \omega$, $s_{n,i}, s_t \in 2^{<\omega}$ recursively for all $n \in \omega$, $i \in \{0,1\}$ and $t \in 2^{<\omega}$ satisfying

(1) $q_0 = 0$ and $q_{n+1} = q_n + \mathrm{lh}(s_{n,0})$.

(2) $s_{0,i} = \emptyset$ and $\mathrm{lh}(s_{n,i}) = \mathrm{lh}(s_{n,1-i}) > 0$ when $n > 0$.

(3) $s_\emptyset = \emptyset$ and $s_{t^\frown i} = s_t{}^\frown s_{\mathrm{lh}(t)+1,i}$ for all $t \in 2^{<\omega}$, $i \in \{0,1\}$.

(4) $\frac{1}{n+1} \leq \sum_{k=0}^{\mathrm{lh}(s_{n+1,0})} \frac{|s_{n+1,0}(k)-s_{n+1,1}(k)|}{q_n+k+1} \leq \frac{2}{n+1}$.

(5) $N_{s_t} \subseteq A$.

(6) If $t \in 2^n$ then $\mu(A_{(s_t)}) > 1 - 2^{-n}$.

Suppose this can be done. We claim that the map $2^\omega \to A : x \mapsto a_x$ defined by

$$a_x = \bigcup_{n\in\omega} s_{x\restriction n}$$

is a Borel (in fact, continuous) reduction of E_I to $E_I \restriction A$. To see this, fix $x, y \in 2^\omega$ and note that by (4) we have that

$$\sum_{n=0}^{\infty} \frac{|x(n)-y(n)|}{n+1} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\mathrm{lh}(s_{n+1,0})} \frac{|s_{n+1,x(i)}(k) - s_{n+1,y(i)}(k)|}{q_n+k+1} = \sum_{n=0}^{\infty} \frac{|a_x(n)-a_y(n)|}{n+1} \leq 2\sum_{n=0}^{\infty} \frac{|x(n)-y(n)|}{n+1}$$

so that xE_Iy if and only if $a_xE_Ia_y$.

We now show that we can construct a scheme satisfying (1)–(6) above. Suppose q_k, $s_{k,i}$ and s_t have been defined for all $k \leq n$ and $t \in 2^{\leq n}$. It is enough to define $s_{n+1,i}$ satisfying (4)–(6). Define

$$f_{q_n} : 2^\omega \to [0,\infty] : f_{q_n}(x) = \sum_{k=0}^{\infty} \frac{x(k)}{q_n + k + 1}.$$

It is clear that $f_{q_n}(N_{0^k})$ is dense in $[0,\infty]$ for all $k \in \omega$. Let

$$A' = \{x \in A : \lim_{k\to\infty} \mu(A_{(x \restriction k)}) \to 1\},$$

i.e, the set of points in A of density 1. By the Lebesgue density theorem [9, 17.9] we have $\mu(A \setminus A') = 0$. Let $A'' = \bigcap_{t \in 2^n} A'_{(s_t)}$ and note that by (6) we have $\mu(A'') > 0$. Thus the set of differences $A'' - A''$ contains a neighborhood of 0^∞ by [9, 17.13]. It follows that there are $x_0, x_1 \in A''$ such that

$$\frac{1}{n+2} \leq \sum_{k=0}^{\infty} \frac{|x_0(k) - x_1(k)|}{q_n + k + 1} \leq \frac{2}{n+2}.$$

Since all points in $A'_{(s_t)}$ have density 1 in $A'_{(s_t)}$ there is some $k_0 \in \omega$ such that

$$\mu(A'_{(s_t^\frown x_i \restriction k_0)}) > 1 - 2^{-n-1}$$

for all $t \in 2^n$. Defining $s_{n+1,i} = x_i \restriction k_0$, it is then clear that (4)–(6) holds. $\square$

Proof of Proposition 1 As the proof easily relativizes, assume that $a = 0$. We proceed exactly as in [3, Proposition 4.2]. Suppose $A \subseteq P(2^\omega)$ is a Σ^1_2 m.o. family. Recall from [10] and [3, p. 1406] that there is a Borel function $2^\omega \to P(2^\omega) : x \mapsto \mu^x$ such that

$$x E_I y \Longrightarrow \mu^x \approx \mu^y$$

and

$$x \not\!E_I y \Longrightarrow \mu^x \perp \mu^y.$$

Define as in [3, Proposition 4.2] a relation $Q \subseteq 2^\omega \times P(2^\omega)^\omega$ by

$$Q(x, (\nu_n)) \iff (\forall n)(\nu_n \in A \wedge \nu_n \not\perp \mu^x) \wedge (\forall \mu)(\mu \not\perp \mu^x \longrightarrow (\exists n)\nu_n \not\perp \mu)$$

and note that this is Σ^1_2 when A is. Note that $Q(x, (\nu_n))$ precisely when (ν_n) enumerates the measures in A not orthogonal to μ^x (this set is always countable, see [10, Theorem 3.1].) Since A is maximal, each section Q_x is non-empty, and so we can uniformize Q with a (total) function $f : 2^\omega \to p(2^\omega)^\omega$ having a Δ^1_2 graph. Note that assignment

$$x \mapsto A(x) = \{f(x)_n : n \in \mathcal{N}\}$$

is invariant on the E_I classes.

If there is a Cohen real over L it follows from [6] that f is Baire measurable. Since E_I is a turbulent equivalence relation (in the sense of Hjorth, see e.g. [10]) the map $x \mapsto A(x)$ must be constant on a comeagre set. But this contradicts that all E_I classes are meagre.

If on the other hand there is a random real over L, then f is Lebesgue measurable by [6]. Let $F \subseteq 2^\omega$ be a closed set with positive measure on which f is continuous, and let $g : 2^\omega \to F$ be a Borel reduction of E_I to $E_I \restriction F$. Note that $x \mapsto A(g(x))$ is then an E_I-invariant Borel assignment of countable subsets of $p(2^\omega)$, and so since E_I is turbulent the function $f \circ g$ must be constant on a comeagre set. This again contradicts that all E_I classes are meagre. $\square$

3 Δ^1_3 w.o. of the reals, Π^1_2 m.o. family, no $\boldsymbol{\Sigma}^1_2$ m.o. families with $\mathfrak{b} = \mathfrak{c} = \omega_3$

We proceed with the proof of Theorem 1. We will use a modification of the model constructed in [2]. The preliminary stage $\mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ of the iteration will coincide almost identically with the preliminary stage $\mathbb{P}_0$ of [2] (see Step 0 through Step 2). For convenience of the reader we outline its construction. We work over the constructible universe L.

Recall that a transitive ZF^- model is *suitable* if $\omega_3^{\mathcal{M}}$ exists and $\omega_3^{\mathcal{M}} = \omega_3^{L^{\mathcal{M}}}$. If $\mathcal{M}$ is suitable then also $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$ and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

Fix a $\diamondsuit_{\omega_2}(\mathrm{cof}(\omega_1))$ sequence $\langle G_\xi : \xi \in \omega_2 \cap \mathrm{cof}(\omega_1)\rangle$ which is Σ_1-definable over L_{ω_2}. For $\alpha < \omega_3$, let W_α be the L-least subset of ω_2 coding α and let $S_\alpha = \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) : G_\xi = W_\alpha \cap \xi \neq \emptyset\}$. Then $\vec{S} = \langle S_\alpha : 1 < \alpha < \omega_3\rangle$ is a sequence of stationary subsets of $\omega_2 \cap \mathrm{cof}(\omega_1)$, which are mutually almost disjoint.

For every α such that $\omega \leq \alpha < \omega_3$ shoot a club C_α disjoint from S_α via the poset $\mathbb{P}^0_\alpha$, consisting of all closed subsets of ω_2 which are disjoint from S_α with the extension relation being end-extension, and let $\mathbb{P}^0 = \prod_{\alpha<\omega_3} \mathbb{P}^0_\alpha$ be the direct product of the $\mathbb{P}^0_\alpha$'s with supports of size ω_1, where for $\alpha \in \omega$, $\mathbb{P}^0_\alpha$ is the trivial poset. Then $\mathbb{P}^0$ is countably closed, ω_2-distributive and ω_3-c.c.

For every α such that $\omega \leq \alpha < \omega_3$ let $D_\alpha \subseteq \omega_3$ be a set coding the triple $\langle C_\alpha, W_\alpha, W_\gamma\rangle$ where γ is the largest limit ordinal $\leq \alpha$. Let

$$E_\alpha = \{\mathcal{M} \cap \omega_2 : \mathcal{M} \prec L_{\alpha+\omega_2+1}[D_\alpha], \omega_1 \cup \{D_\alpha\} \subseteq \mathcal{M}\}.$$

Then E_α is a club on ω_2. Choose $Z_\alpha \subseteq \omega_2$ such that $Even(Z_\alpha) = D_\alpha$, where $Even(Z_\alpha) = \{\beta : 2 \cdot \beta \in Z_\alpha\}$, and if $\beta < \omega_2$ is the $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_\alpha \cap \beta \in \mathcal{M}$, then $\beta \in E_\alpha$. Then we have:

$(*)_\alpha$: If $\beta < \omega_2$, $\mathcal{M}$ is a suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_\alpha \cap \beta \in \mathcal{M}$, then $\mathcal{M} \models \psi(\omega_2, Z_\alpha \cap \beta)$, where $\psi(\omega_2, X)$ is the formula "*Even*(X) codes a triple $\langle \bar{C}, \bar{W}, \bar{\bar{W}}\rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Similarly to $\vec{S}$ define a sequence $\vec{A} = \langle A_\xi : \xi < \omega_2 \rangle$ of stationary subsets of ω_1 using the "standard" $\diamondsuit$-sequence. Code Z_α by a subset X_α of ω_1 with the poset $\mathbb{P}^1_\alpha$ consisting of all pairs $\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$ and $t_0 \backslash s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$. Then X_α satisfies the following condition:

$(**)_\alpha$: If $\omega_1 < \beta \leq \omega_2$ and $\mathcal{M}$ is a suitable model such that $\omega_2^{\mathcal{M}} = \beta$ and $\{X_\alpha\} \cup \omega_1 \subset \mathcal{M}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_\alpha)$, where $\phi(\omega_1, \omega_2, X)$ is the formula: " Using the sequence $\vec{A}$, X almost disjointly codes a subset $\bar{Z}$ of ω_2, such that $Even(\bar{Z})$ codes a triple $\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Let $\mathbb{P}^1 = \prod_{\alpha < \omega_3} \mathbb{P}^1_\alpha$, where $\mathbb{P}^1_\alpha$ is the trivial poset for all $\alpha \in \omega$, with countable support. Then $\mathbb{P}^1$ is countably closed and has the ω_2-c.c.

Finally we force a localization of the X_α's. Fix ϕ as in $(**)_\alpha$ and let $\mathcal{L}(X, X')$ be the poset defined in [2, Definition 1], where $X, X' \subset \omega_1$ are such that $\phi(\omega_1, \omega_2, X)$ and $\phi(\omega_1, \omega_2, X')$ hold in any suitable model $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$ containing X and X', respectively. That is $\mathcal{L}(X, X')$ consists of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

(1) if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$

(2) if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$

(3) if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \restriction \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X \cap \gamma) \wedge \phi(\omega_1, \omega_2, X' \cap \gamma)$.

The extension relation is end-extension. Then let $\mathbb{P}^2_{\alpha+m} = \mathcal{L}(X_{\alpha+m}, X_\alpha)$ for every $\alpha \in Lim(\omega_3) \backslash \{0\}$ and $m \in \omega$. Let $\mathbb{P}^2_{\alpha+m}$ be the trivial poset for $\alpha = 0$, $m \in \omega$ and let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. Note that the poset $\mathbb{P}^2_{\alpha+m}$, where $\alpha > 0$, produces a generic function in ${}^{\omega_1}2$ (of $L^{\mathbb{P}^0 * \mathbb{P}^1}$), which is the characteristic function of a subset $Y_{\alpha+m}$ of ω_1 with the following property:

$(***)_\alpha$: For every $\beta < \omega_1$ and any suitable $\mathcal{M}$ such that $\omega_1^{\mathcal{M}} = \beta$ and $Y_{\alpha+m} \cap \beta$ belongs to $\mathcal{M}$, we have $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \phi(\omega_1, \omega_2, X_\alpha \cap \beta)$.

Claim $\mathbb{P}_0 := \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is ω-distributive.

Proof [2, Lemma 1]. □

Let $\vec{B} = \langle B_{\zeta,m} : \zeta < \omega_1, m \in \omega\rangle$ be a nicely definable sequence of almost disjoint subsets of ω. We will define a finite support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_3, \beta < \omega_3\rangle$ such that $\mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$, for every $\alpha < \omega_3$, $\mathbb{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset, in $L^{\mathbb{P}_{\omega_3}}$ there is a Δ^1_3-definable wellorder of the reals, a Π^1_2-definable maximal family of orthogonal measures and there are no Σ^1_2-definable maximal families of orthogonal measures. Along the iteration for every $\alpha < \omega_3$, we will define in $V^{\mathbb{P}_\alpha}$ a set O_α of orthogonal measures and for $\alpha \in Lim(\alpha)$ a subset A_α of $[\alpha, \alpha+\omega)$. Every $\mathbb{Q}_\alpha$ will add a generic real, whose $\mathbb{P}_\alpha$-name will be denoted u_α and similarly to the proof of [2, Lemma 2] one can prove that $L[G_\alpha] \cap {}^\omega\omega = L[\langle u_\xi^{G_\alpha} : \xi < \alpha\rangle] \cap {}^\omega\omega$ for every $\mathbb{P}_\alpha$-generic filter G_α. This gives a canonical wellorder of the reals in $L[G_\alpha]$ which depends only on the sequence $\langle u_\xi : \xi < \alpha\rangle$, whose $\mathbb{P}_\alpha$-name will be denoted by $<_\alpha$. We can additionally arrange that for $\alpha < \beta$, $<_\alpha$ is an initial segment of $<_\beta$, where $<_\alpha = <_\alpha^{G_\alpha}$ and $<_\beta = <_\beta^{G_\beta}$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter over L, then $<^G = \bigcup\{<_\alpha^G : \alpha < \omega_3\}$ will be the desired wellorder of the reals and $O = \bigcup_{\alpha<\omega_3} O_\alpha$ will be the Π^1_2-definable maximal family of orthogonal measures.

We proceed with the recursive definition of $\mathbb{P}_{\omega_3}$. For every $\nu \in [\omega_2, \omega_3)$ let $i_\nu : \nu \cup \{\langle \xi, \eta\rangle : \xi < \eta < \nu\} \to Lim(\omega_3)$ be a fixed bijection. If G_α is a $\mathbb{P}_\alpha$-generic filter over L, $<_\alpha = <_\alpha^{G_\alpha}$ and x, y are reals in $L[G_\alpha]$ such that $x <_\alpha y$, let $x * y = \{2n : n \in x\} \cup \{2n+1 : n \in y\}$ and $\Delta(x * y) = \{2n+2 : n \in x * y\} \cup \{2n+1 : n \notin x * y\}$. Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter G_α.

If $\alpha = \omega_2 \cdot \alpha' + \xi$, where $\alpha' > 0$, $\xi \in Lim(\omega_2)$, let $\nu = o.t.(<^{G_\alpha}_{\omega_2\cdot\alpha'})$ and let $i = i_\nu$.

Case 1. If $i^{-1}(\xi) = \langle \xi_0, \xi_1\rangle$ for some $\xi_0 < \xi_1 < \nu$, let x_{ξ_0} and x_{ξ_1} be the ξ_0-th and ξ_1-th reals in $L[G_{\omega_2\cdot\alpha'}]$ according to the wellorder $<^{G_\alpha}_{\omega_2\cdot\alpha'}$. In $L^{\mathbb{P}_\alpha}$ let

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m\in\Delta(x_{\xi_0} * x_{\xi_1})} Y_{\alpha+m} \times \{m\}]^{<\omega}\},$$

where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if $s_1 \subseteq t_1$, s_0 is an initial segment of t_0 and $(t_0 \backslash s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m\rangle \in s_1$. Let u_α be the generic real added by $\mathbb{Q}_\alpha$, $A_\alpha = \alpha + \omega \backslash \Delta(x_{\xi_0} * x_{\xi_1})$ and $O_\alpha = \emptyset$.

Case 2. Suppose $i^{-1}(\xi) = \zeta \in \nu$. If the ζ-th real according to the wellorder $<^{G_\alpha}_{\omega_2\cdot\alpha'}$ is not the code of a measure orthogonal to $O'_\alpha = \bigcup_{\gamma<\alpha} O_\gamma$, let $\mathbb{Q}_\alpha$ be the trivial poset, $A_\alpha = \emptyset$, $O_\alpha = \emptyset$. Otherwise, i.e. in case x_ζ is a code for a measure orthogonal to O'_α, let

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m\in\Delta(x_\zeta)} Y_{\alpha+m} \times \{m\}]^{<\omega}\},$$

where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if $s_1 \subseteq t_1$, s_0 is an initial segment of t_0 and $(t_0 \backslash s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m\rangle \in s_1$. Let u_α be the generic real added by $\mathbb{Q}_\alpha$. In $L^{\mathbb{P}_{\alpha+1}} = L^{\mathbb{P}_\alpha * \mathbb{Q}_\alpha}$ let $g_\alpha = G(x_\zeta, u_\alpha)$ be the code of a measure equivalent to μ_{x_ζ} which codes u_α (see [3, Lemma 3.5]) and let $O_\alpha = \{\mu_{g_\alpha}\}$. Let $A_\alpha = \alpha + \omega \backslash \Delta(u_\alpha)$.

If α is not of the above form, i.e. α is a successor or $\alpha \in \omega_2$, let $\mathbb{Q}_\alpha$ be the following poset for adding a dominating real:

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in \omega^{<\omega}, s_1 \in [\text{o.t.}(<_\alpha^{G_\alpha})]^{<\omega}\},$$

where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if s_0 is an initial segment of t_0, $s_1 \subseteq t_1$, and $t_0(n) > x_\xi(n)$ for all $n \in \text{dom}(t_0)\backslash\text{dom}(s_0)$ and $\xi \in s_1$, where x_ξ is the ξ-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $<_\alpha^{G_\alpha}$. Let $A_\alpha = \emptyset$, $O_\alpha = \emptyset$.

With this the definition of $\mathbb{P}_{\omega_3}$ is complete. Let $O = \bigcup_{\alpha<\omega_3} O_\alpha$. In $L^{\mathbb{P}_{\omega_3}}$ we have: *ν is a measure in the set O if and only if for every countable suitable model $\mathcal{M}$ such that $\nu \in \mathcal{M}$, there is $\bar{\alpha} < \omega_3^{\mathcal{M}}$ such that $S_{\bar{\alpha}+m}$ is nonstationary in $(L[r(\nu)])^{\mathcal{M}}$ for every $m \in \Delta(r(\nu))$.* Therefore O has indeed a Π^1_2 definition. Furthermore O is maximal in $P_c(2^\omega)$. Indeed, suppose in $L^{\mathbb{P}_{\omega_3}}$ there is a code x for a measure orthogonal to every measure in the family O. Choose α minimal such that $\alpha = \omega_2 \cdot \alpha' + \xi$ for some $\alpha' > 0$ and $\xi \in Lim(\omega_2)$ and $x \in L[G_{\omega_2\cdot\alpha'}]$. Let $\nu = o.t.(<^{G_\alpha}_{\omega_2\cdot\alpha'})$ and let $i = i_\nu$. Then $x = x_\zeta$ is the ζ-th real according to the wellorder $<^{G_\alpha}_{\omega_2\cdot\alpha'}$, where $\zeta \in \nu$ and so for some $\xi \in Lim(\omega_2)$, $i^{-1}(\xi) = \zeta$. But then $x_\zeta = x$ is the code of a measure orthogonal to O_α and so by construction $O_{\alpha+1}$ contains a measure equivalent to μ_x, which is a contradiction. To obtain a Π^1_2-definable m.o. family in $L\mathbb{P}_{\omega_3}$ consider the union of O with the set of all point measures. Just as in [2] one can show that $<$ is indeed a Δ^1_3-definable wellorder of the reals.

Since $\mathbb{P}_{\omega_3}$ is a finite support iteration, we have added Cohen reals along the iteration cofinally often. Thus for every real a in $L^{\mathbb{P}_{\omega_3}}$ there is a Cohen real over $L[a]$ and so by Proposition 1 in $L^{\mathbb{P}_{\omega_3}}$ there are no $\mathbf{\Sigma}^1_2$ m.o. families. Also note that since cofinally often we have added dominating reals, $L^{\mathbb{P}_{\omega_3}} \vDash \mathfrak{b} = \omega_3$.

4 Δ^1_3 w.o. of the reals, a Π^1_2 m.o. family, no $\mathbf{\Sigma}^1_2$ m.o. families with $\mathfrak{c} = \omega_2$

In this section we establish the proof of Theorem 2. The model is obtained as a slight modification of the iteration construction developed in [1]. We restate the definitions of the posets used in this construction. For a more detailed account of their properties see [1]. We work over the constructible universe L.

If $S \subseteq \omega_1$ is a stationary, co-stationary set, then by $Q(S)$ denote the poset of all countable closed subsets of $\omega_1\backslash S$ with the extension relation being end-extension. Recall that $Q(S)$ is $\omega_1\backslash S$-proper, ω-distributive and adds a club disjoint from S (see [1], [5]). For the proof of Theorem 2 we use the form of localization defined in [1, Definition 1]. That is, if $X \subseteq \omega_1$ and $\phi(\omega_1, X)$ is a Σ_1-sentence with parameters ω_1, X which is true in all suitable models containing ω_1 and X as elements, then $\mathcal{L}(\phi)$ be the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal, such that

(1) if $\gamma < |r|$ then $\gamma \in X$ iff $r(2\gamma) = 1$

(2) if $\gamma \leq |r|$, $\mathcal{M}$ is a countable, suitable model containing $r \restriction \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\phi(\gamma, X \cap \gamma)$ holds in $\mathcal{M}$.

The extension relation is end-extension. Recall that $\mathcal{L}(\phi)$ has a countably closed dense subset (see [1, Remark 2]) and that if G is $\mathcal{L}(\phi)$-generic and $\mathcal{M}$ is a countable suitable model containing $(\bigcup G) \restriction \gamma$ as an element, where $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi(\gamma, X \cap \gamma)$ (see [1, Lemma 2]).

We will use also the coding with perfect trees defined in [1, Definition 2]. Let $Y \subseteq \omega_1$ be generic over L such that in $L[Y]$ cofinalities have not been changed and let $\bar{\mu} = \{\mu_i\}_{i \in \omega_1}$ be a sequence of L-countable ordinals such that μ_i is the least $\mu > \sup_{j<i} \mu_j$, $L_\mu[Y \cap i] \vDash ZF^-$ and $L_\mu \vDash$ ω is the largest cardinal. Say that a real R *codes* Y below i if for all $j < i$, $j \in Y$ if and only if $L_{\mu_j}[Y \cap j, R] \vDash ZF^-$. For $T \subseteq 2^{<\omega}$ a perfect tree, let $|T|$ be the least i such that $T \in L_{\mu_i}[Y \cap i]$. Then $\mathcal{C}(Y)$ is the poset of all perfect trees T such that R codes Y below $|T|$, whenever R is a branch through T, where for T_0, T_1 conditions in $\mathcal{C}(Y)$, $T_0 \leq T_1$ if and only if T_0 is a subtree of T_1. Recall also that $\mathcal{C}(Y)$ is proper and ${}^\omega\omega$-bounding (see [1, Lemmas 7,8]).

Fix a bookkeeping function $F : \omega_2 \to L_{\omega_2}$ and a sequence $\vec{S} = (S_\beta : \beta < \omega_2)$ of almost disjoint stationary subsets of ω_1, defined as in [1, Lemma 14]. Thus F and $\vec{S}$ are Σ_1-definable over L_{ω_2} with parameter ω_1, $F^{-1}(a)$ is unbounded in ω_2 for every $a \in L_{\omega_2}$ and whenever $\mathcal{M}, \mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$ then $F^{\mathcal{M}}, \vec{S}^{\mathcal{M}}$ agree with $F^{\mathcal{N}}$, $\vec{S}^{\mathcal{N}}$ on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. Also if $\mathcal{M}$ is suitable and $\omega_1^{\mathcal{M}} = \omega_1$ then $F^{\mathcal{M}}, \bar{S}^{\mathcal{M}}$ equal the restrictions of F, $\vec{S}$ to the ω_2 of $\mathcal{M}$. Fix also a stationary subset S of ω_1 which is almost disjoint from every element of $\vec{S}$.

Recursively we will define a countable support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ and a sequence $\langle O_\alpha : \alpha \in \omega_2 \rangle$, such that in $L^{\mathbb{P}_{\omega_2}}$ there is a Δ^1_3-definable wellorder of the reals and $O = \bigcup_{\alpha < \omega_2} O_\alpha$ is a maximal family of orthogonal measures. Define the wellorder $<_\alpha$ in $L[G_\alpha]$ where G_α is $\mathbb{P}_\alpha$-generic just as in [1]. We can assume that all names for reals are nice and that for $\alpha < \beta < \omega_2$, all $\mathbb{P}_\alpha$-names for reals precede in the canonical wellorder $<_L$ of L all $\mathbb{P}_\beta$-names for reals, which are not $\mathbb{P}_\alpha$-names. For each $\alpha < \omega_2$, define a wellorder $<_\alpha$ on the reals of $L[G_\alpha]$, where G_α is a $\mathbb{P}_\alpha$-generic as follows. If x is a real in $L[G_\alpha]$ let σ_x^α be the $<_L$-least $\mathbb{P}_\gamma$-name for x, where $\gamma \leq \alpha$ is least so that x has a $\mathbb{P}_\gamma$-name. For x, y reals in $L[G_\alpha]$ define $x <_\alpha y$ if and only if $\sigma_x^\alpha <_L \sigma_y^\alpha$. Note that whenever $\alpha < \beta$, then $<_\alpha$ is an initial segment of $<_\beta$.

We proceed with the definition of the poset. Let $\mathbb{P}_0$ be the trivial poset. Suppose $\mathbb{P}_\alpha$ and $\langle O_\gamma : \gamma < \alpha \rangle$ have been defined. Let $\mathbb{Q}_\alpha = \mathbb{Q}_\alpha^0 * \mathbb{Q}_\alpha^1$ be a $\mathbb{P}_\alpha$-name for a poset where $\mathbb{Q}_\alpha^0$ is a $\mathbb{P}_\alpha$-name for the random real forcing and $\mathbb{Q}_\alpha^1$ is defined as follows:

Case 1. If $F(\alpha) = \{\sigma_x^\alpha, \sigma_y^\alpha\}$ for some pair of reals x, y in $L[G_\alpha]$, then define $\mathbb{Q}_\alpha$ as in [1]. That is $\mathbb{Q}_\alpha$ is a three stage iteration $\mathbb{K}_\alpha^0 * \mathbb{K}_\alpha^1 * \mathbb{K}_\alpha^2$ where:

(1) In $V^{\mathbb{P}_\alpha * \mathbb{Q}_\alpha^0}$, $\mathbb{K}_\alpha^0$ is the direct limit $\langle \mathbb{P}_{\alpha,n}^0, \mathbb{K}_{\alpha,n}^0 : n \in \omega \rangle$, where $\mathbb{K}_{\alpha,n}^0$ is a $\mathbb{P}_{\alpha,n}^0$-name for $Q(S_{\alpha+2n})$ for $n \in x_\alpha * y_\alpha$, and $\mathbb{K}_{\alpha,n}^0$ is a $\mathbb{P}_{\alpha,n}^0$-name for $Q(S_{\alpha+2n+1})$ for $n \notin x_\alpha * y_\alpha$.

(2) Let G^0_α be a $\mathbb{P}_\alpha * \mathbb{Q}^0_\alpha$-generic filter and let H_α be a $\mathbb{K}^0_\alpha$-generic over $L[G^0_\alpha]$. In $L[G^0_\alpha * H_\alpha]$ let X_α be a subset of ω_1 coding α, coding the pair (x_α, y_α), coding a level of L in which α has size at most ω_1 and coding the generic $G^0_\alpha * H_\alpha$, which we can regard as a subset of an element of L_{ω_2}. Let $\mathbb{K}^1_\alpha = \mathcal{L}(\phi_\alpha)$ where $\phi_\alpha = \phi_\alpha(\omega_1, X)$ is the Σ_1-sentence which holds if and only if X codes an ordinal $\bar{\alpha} < \omega_2$ and a pair (x, y) such that $S_{\bar{\alpha}+2n}$ is nonstationary for $n \in x * y$ and $S_{\bar{\alpha}+2n+1}$ is nonstationary for $n \notin x * y$. Let X_α be a $\mathbb{P}^0_\alpha * \mathbb{Q}^0_\alpha * \mathbb{K}^0_\alpha$-name for X_α and let $\mathbb{K}^1_\alpha$ be a $\mathbb{P}^0_\alpha * \mathbb{Q}^0_\alpha * \mathbb{K}^0_\alpha$-name for $\mathbb{K}^1_\alpha$.

(3) Let Y_α be $\mathbb{K}^1_\alpha$-generic over $L[G^0_\alpha * H_\alpha]$. Note that the even part of Y_α-codes X_α and so codes the generic $G^0_\alpha * H_\alpha$. Then in $L[Y_\alpha] = L[G^0_\alpha * H_\alpha * Y_\alpha]$, let $\mathbb{K}^2_\alpha = \mathcal{C}(Y_\alpha)$. Finally, let $\mathbb{K}^2_\alpha$ be a $\mathbb{P}_\alpha * \mathbb{Q}^0_\alpha * \mathbb{K}^0_\alpha * \mathbb{K}^1_\alpha$-name for $\mathbb{K}^2_\alpha$.

Case 2. If $F(\alpha) = \{\sigma^\alpha_x\}$ where x is a code for a measure orthogonal to $\bigcup_{\gamma<\alpha} O_\gamma$, then let $\mathbb{Q}^1_\alpha$ be a $\mathbb{P}_\alpha * \mathbb{Q}^1_\alpha$-name for $\mathbb{K}^0_\alpha * \mathbb{K}^1_\alpha * \mathbb{K}^2_\alpha$ where in $L^{\mathbb{P}_\alpha * \mathbb{Q}_\alpha}$, $\mathbb{K}^0_\alpha$ is the direct limit $\langle \mathbb{P}^0_{\alpha,n}, \mathbb{Q}^0_{\alpha,n} : n \in \omega\rangle$ where $\mathbb{Q}^0_{\alpha,n}$ is a $\mathbb{P}^0_{\alpha,n}$-name for $Q(S_{\alpha+2n})$ for every $n \in x$ and a $\mathbb{P}^0_{\alpha,n}$-name for $Q(S_{\alpha+2n+1})$ for every $n \notin x$. Define $\mathbb{K}^1_\alpha$ and $\mathbb{K}^2_\alpha$ just as in *Case 1*. In $L^{\mathbb{P}_\alpha * \mathbb{Q}_\alpha}$ let $g = G(x, R_\alpha)$ be a code for a measure which is equivalent to μ_x and codes the real R_α. Let $O_\alpha = \{\mu_g\}$.

In any other case, let $\mathbb{Q}_\alpha$ be a $\mathbb{P}_\alpha$-name for the trivial poset, $O_\alpha = \emptyset$. With this the definition of $\mathbb{P}_{\omega_2}$ and the family $O = \bigcup_{\gamma<\omega_2} O_\alpha$ is complete.

Claim $O = \bigcup_{\gamma<\omega_2} O_\gamma$ is a maximal family of orthogonal measures in $P_c(2^\omega)$.

Proof It is clear that O is a family of orthogonal measures. It remains to verify its maximality. Suppose the contrary and let f be a code for a measure in $L[G]$ where G is $\mathbb{P}_{\omega_3}$-generic over L, which is orthogonal to all measures in O. Fix α minimal such that f is in $L[G_\alpha]$ and let σ be the $<_L$-least name for f. Since $F^{-1}(\sigma)$ is unbounded, there is $\beta \geq \alpha$ such that $F(\beta) = \{\sigma\}$. Therefore $\mathbb{Q}_\beta$ is nontrivial and $O_\beta = \{\mu_g\}$ for some measure μ_g which is equivalent to μ_f, which is a contradiction. □

Clearly, $\mu \in O$ *if and only if for every countable suitable model* $\mathcal{M}$ *such that* $\mu \in \mathcal{M}$ *there is* $\alpha < \omega_2^{\mathcal{M}}$ *such that* $S_{\alpha+m}$ *is nonstationary in* $L[r(\mu)]^{\mathcal{M}}$ *for every* $m \in \Delta(r(\mu))$. Thus our family O has indeed a Π^1_2 definition. Just as in the proof of Theorem 1, to obtain a Π^1_2-definable m.o. family in $L^{\mathbb{P}_{\omega_3}}$ consider the union of O with the set of all point measures.

Since for every real $a \in L^{\mathbb{P}_{\omega_3}}$ there is a random real over L, by Proposition 1 in $L^{\mathbb{P}_{\omega_3}}$ there are no $\mathbf{\Sigma}^1_2$ m.o. families. The bounding number $\mathfrak{b}$ remains ω_1 in $L^{\mathbb{P}_{\omega_3}}$, since the countable support iteration of S-proper ${}^\omega\omega$-bounding posets is ${}^\omega\omega$-bounding (see [1, Lemma 18] or [5]). □

Remark 4.1 In [3] the following question was raised:

Question 1 If there is a Π^1_1 m.o. family, are all reals constructible?

This is to our knowledge still unsolved. Törnquist has recently shown that the existence of a Σ^1_2 m.o. family implies the existence of a Π^1_1 m.o. family, and that the existence of Σ^1_2 mad family implies the existence of a Π^1_1 mad family.

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